The Area of ℓ-r Flakes

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1 Introduction

The purpose of this paper is to explore the possibility of a more generalized form of the well-known Koch Snowflake.

2 Abstract

An ℓ-r flake is a polygon with infinite perimeter but sometimes finite area, constructed by performing an infinite set of transformations on an equilateral triangle, by projecting new equilateral triangles on every side of the shape at a given position. Parameters ℓ and r are defined as the ratios between the left and right adjacent sides to the side of a projection itself (illustrated in Figure 2).

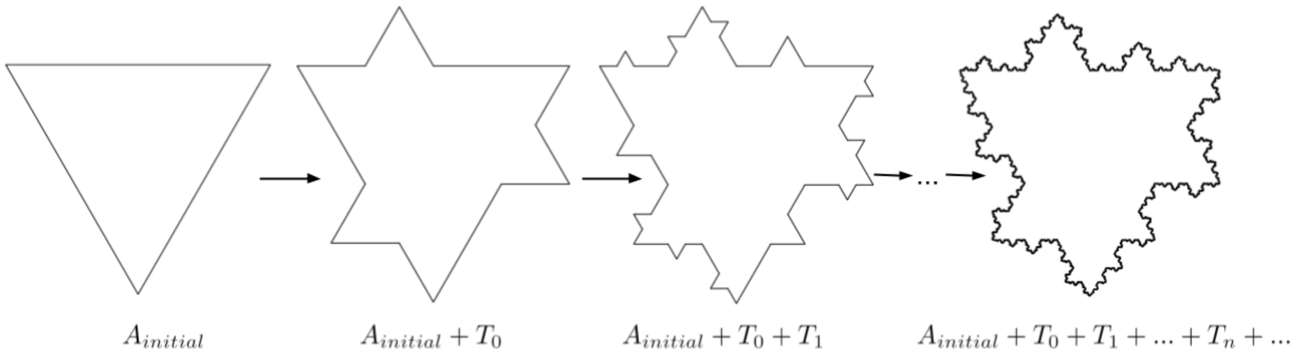


Figure 1: The process of constructing an ℓ-r flake

The ℓ-r flake is a generalization of the Koch Snowflake fractal, where instead of projecting a new equilateral triangle from the middle third of each side, the projection's position and size relative to its parent side are determined by the ℓ and r parameters. The Koch Snowflake is in fact an instance of an ℓ-r flake where both ℓ and r are equal to 1.

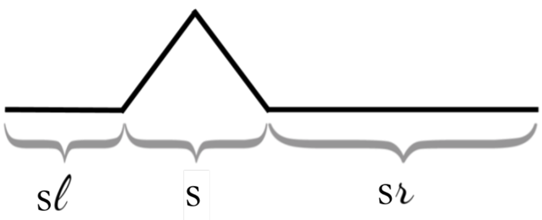


Figure 2: Illustration of left and right ratios

Definition 1. ***projection*** a new equilateral triangle that is constructed off of an existing side, related by some proportion to that side length.

Definition 2. ***transformation*** the act of taking an existing ℓ-r flake and adding new projections to every distinct side.

It is important to note that the area of the final infinite fractal is equivalent to an infinite sum of triangular areas, as the polygon is constructed entirely out of equilateral triangles.

The final area itself can be expressed as a series, starting from an initial area and infinitely adding new area at each transformation which proceeds to get smaller and smaller.

$$\begin{aligned} A_{flake} &= A_{initial} + T_0 + T_1 + ... + T_n + ... \\ &= A_{initial} + \sum_{n=0}^{\infty} T_n \end{aligned} \tag{1}$$

where $A_{initial}$ is the area of the starting triangle, and T_n is the total area added at the n^{th} transformation.

Under certain conditions, the area formed by starting with an equilateral triangle and performing an infinite series of transformations converges. The objective of this paper is to explain the ratio between the convergent area and the initial triangle's area.

Theorem 1. The ratio between the final area of a convergent ℓ-r flake and its initial triangular area is

$$\frac{(\ell+1)(r+1)}{(\ell+1)(r+1)-\frac{3}{2}} \tag{2}$$

for any positive ℓ, r where $\ell + r + r\ell > \frac{1}{2}$.

3 Proof: Ratio of Convergent Area

Along with the definitions of ℓ and r, we will define k as the ratio between a given side length and the side length of the triangle that projects directly from it. That is,

$$\begin{aligned} sk &= s\ell + sr + s \\ sk &= s(\ell + r + 1) \\ k &= \ell + r + 1 \end{aligned} \tag{3}$$

where s is the side length of any given projection.

Before we can examine the various ways projections can occur, let's understand how the areas relate between two equilateral triangles, one with side length a fraction of the other.

3.1 Relating the areas of proportional equilateral triangles

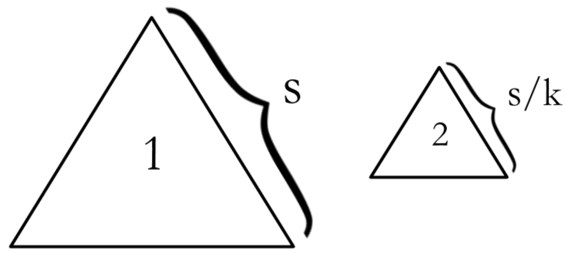


Figure 3: Comparison of proportional equilateral areas

The areas A_1 and A_2 pictured above are related by some proportion β, which can be found by comparing their areas in terms of side length s.

$$
\begin{aligned}
A_1 &= \beta A_2 \\
s^2 \cdot \frac{\sqrt{3}}{4} &= \beta \cdot \frac{s^2}{k^2} \cdot \frac{\sqrt{3}}{4} \\
1 &= \frac{\beta}{k^2} \\
k^2 &= \beta
\end{aligned} \tag{4}
$$

So we can see that an equilateral with side length $\frac{1}{k}$ of its projecting side will have an area $\frac{1}{k^2}$ of the "parent" equilateral from which it projected.

3.2 Initial transformation

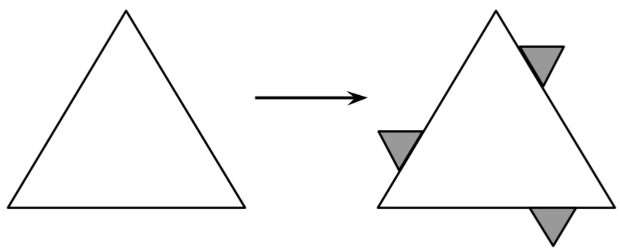


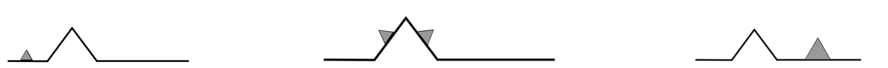
Figure 4: Initial Transformation (T_0)

The initial transformation is unique in that it only involves *one* type of projection. Each child in this transformation has an area $\frac{1}{k^2}$ of the parent, which is deemed 1 in order for our final result to be a ratio, leaving them at $\frac{1}{k^2}$.

From here, any new triangle from any depth of transformation can be related back to the areas of these three initial projections, allowing us to concisely express the total added area with an infinite sum.

3.3 Projection Types

Given an existing projection on a side, which we will refer to as the "parent" projection, there are 3 ways to project, as follows:



(a) Left projection (b) Direct projection (c) Right projection

Figure 5: Projection types

In the case of a direct projection, the area of the child (projection) is simply:

$$
A_{direct} = \frac{1}{k^2} A_p \tag{5}
$$

where A_p is the area of the parent.

In the case of adjacent projections, however, we first have to realize that, while the children do not project directly off of the parent p, they project off of other equilaterals that have areas proportional to A_p. See the figure below, with these equilaterals L and R:

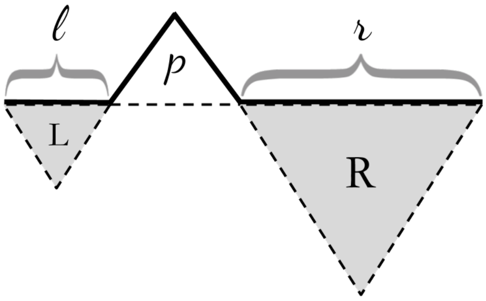


Figure 6: Hidden equilaterals L and R

Since ℓ and r are defined as the proportions between the side length of p and its adjacent sides, i.e. $S_L = \ell S_p$, we know that the side length of triangle p is $\frac{1}{\ell}$ of triangle L, and $\frac{1}{r}$ of triangle R, revealing that the ratios between their areas are ℓ^2 and r^2, respectively (see Section 2.1):

$$\begin{aligned} A_L &= \ell^2 A_p \\ A_R &= r^2 A_p \end{aligned} \tag{6}$$

giving the relative areas of the adjacent projections as:

$$\begin{aligned} A_{left} &= \frac{1}{k^2} A_L = \frac{\ell^2}{k^2} A_p \\ A_{right} &= \frac{1}{k^2} A_R = \frac{r^2}{k^2} A_p \end{aligned} \tag{7}$$

Knowing how to determine the area of any type of projection relative to its parent, and knowing that each parent will beget one left adjacent projection, one right adjacent projection, and 2 direct projections (since 2 faces of the parent projection are showing), we are able to concisely define the area added at any n^{th} transformation.

3.4 Developing an expression for the generalized transformation

Given some projection p, the total area of child projections off of p is given by

$$\begin{aligned} A_c &= A_{left} + (2 \cdot A_{direct}) + A_{right} \\ &= \frac{\ell^2}{k^2} A_p + \frac{2}{k^2} A_p + \frac{r^2}{k^2} A_p \\ &= A_p \left(\frac{\ell^2 + 2 + r^2}{k^2} \right) \end{aligned} \tag{8}$$

Starting from initial transformation T_0, our added area was $\frac{3}{k^2}$, since there were 3 projections of area $\frac{1}{k^2}$ each. These projections act as parents for every following transformation.

At T_1, we add the total *child area* of T_0,

$$T_1 = T_0 \cdot \left(\frac{\ell^2 + 2 + r^2}{k^2} \right) = \frac{3}{k^2} \left(\frac{\ell^2 + 2 + r^2}{k^2} \right) \tag{9}$$

using the predetermined ratio of parent to child area.

As n increases, T_n can be defined recursively as the total child area of its predecessor, T_{n-1}, which is in turn the total child area of *its* predecessor, T_{n-2} and so on, yielding

$$T_n = T_{n-1} \cdot \left(\frac{\ell^2 + 2 + r^2}{k^2} \right) \tag{10}$$

Since this definition is, in essence, repeatedly multiplying the parent-child ratio all the way back to T_0, the same definition can be expressed simply with an exponent:

$$T_n = \frac{3}{k^2}\left(\frac{\ell^2+2+r^2}{k^2}\right)^n \tag{11}$$

3.5 Determining the convergent area

Recalling that for any ℓ-r flake,

$$A_{final} = A_{initial} + \sum_{n=0}^{\infty} T_n \tag{12}$$

we can choose $A_{initial}$ as 1, allowing us to calculate the *ratio* between final and initial, and substitute in our definition for the generalized transformation T_n

$$\begin{aligned} A_{final} &= 1 + \sum_{n=0}^{\infty} \frac{3}{k^2}\left(\frac{\ell^2+2+r^2}{k^2}\right)^n \\ &= 1 + \frac{3}{k^2}\sum_{n=0}^{\infty}\left(\frac{\ell^2+2+r^2}{k^2}\right)^n \end{aligned} \tag{13}$$

Recalling our definition of k as $\ell + r + 1$, we have:

$$1 + \frac{3}{(\ell+r+1)^2}\sum_{n=0}^{\infty}\left(\frac{\ell^2+2+r^2}{(\ell+r+1)^2}\right)^n \tag{14}$$

Our summation has taken the form of a geometric series, with common ratio w as

$$w = \frac{\ell^2+2+r^2}{(\ell+r+1)^2} \tag{15}$$

Note that if $|w| < 1$, the series will converge. If $\ell + r + r\ell > \frac{1}{2}$ (see Appendix A), the following expression holds:

$$(\ell+r+1)^2 > \ell^2 + 2 + r^2 \tag{16}$$

asserting that the series converges. We are then able to simplify (see Appendix B):

$$\begin{aligned} &1 + \frac{3}{(\ell+r+1)^2}\cdot\frac{1}{1-\frac{\ell^2+2+r^2}{(\ell+r+1)^2}} \\ &= 1 + \frac{3}{(\ell+r+1)^2}\cdot\frac{(\ell+r+1)^2}{(\ell+r+1)^2-(\ell^2+2+r^2)} \\ &= 1 + \frac{3}{2\ell+2r+2r\ell-1} \\ &= \frac{\ell+r+r\ell+1}{\ell+r+r\ell-\frac{1}{2}} \end{aligned} \tag{17}$$

Resulting in the final ratio between the final and initial areas for any ℓ-r flake as

$$\frac{(\ell+1)(r+1)}{(\ell+1)(r+1)-\frac{3}{2}} \tag{18}$$

4 Understanding Conditions of Convergence

Once the final expression for area has been reduced to a geometric series, convergence comes down to whether or not the common ratio w is between -1 and 1. The ratio itself is

$$w = \frac{\ell^2 + 2 + r^2}{(\ell + r + 1)^2} \tag{19}$$

but what does this intuitively mean in the context of the problem?

If you'll recall, this w is actually the ratio between total child area and the parent projection that created it (as derived in section 2.4). That is,

$$w = \frac{A_c}{A_p} \tag{20}$$

where A_c is the total child area of a projection and A_p is the area of that projection itself.

From here the intuitive breakdown of the condition of convergence becomes much simpler:

> **If, at every transformation, a parent projection gives way to *more* area ($w > 1$, $A_c > A_p$) than it currently occupies, the total area will just continue to expand and never converge to a single value.**

This is also true if a parent begets the *same* amount of area as it spans. However, if the child area A_c gets smaller and smaller each time a transformation is performed, the resulting total area will converge.

5 Visualizing the Ratio in 3D Space

By treating the relationship between any ℓ, r pair and its ratio of convergence as a function, we can plot the relationship spatially in three dimensions, with each (ℓ, r) pair represented by a point on the x, y plane and their resulting ratio, $z(\ell, r)$, plotted on the z-axis.

Extending the domain to negative ℓ and r values allows us to observe interesting behavior, though these values do not make "sense" in the physical context of the problem, as the measurement of area becomes questionable when negative values are introduced.

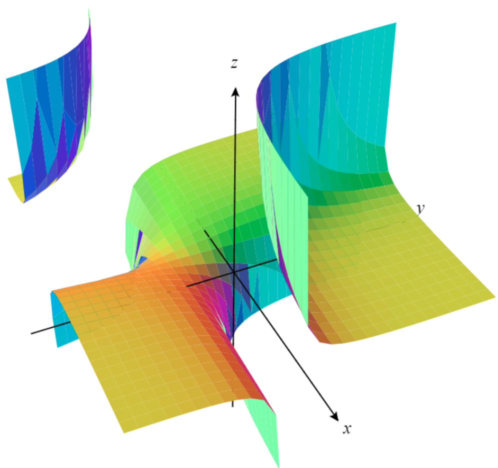


Figure 7: Plot of ratio $z(\ell, r)$

The function has horizontal asymptotes as follows:

$$\begin{aligned} \lim_{\substack{\ell \to \infty \\ r \to \infty}} z(\ell, r) &= \lim_{\substack{\ell \to -\infty \\ r \to -\infty}} z(\ell, r) = 1 \\ \lim_{\substack{\ell \to -\infty \\ r \to \infty}} z(\ell, r) &= \lim_{\substack{\ell \to \infty \\ r \to -\infty}} z(\ell, r) = -1 \end{aligned} \tag{21}$$

as illustrated in Figure 8.

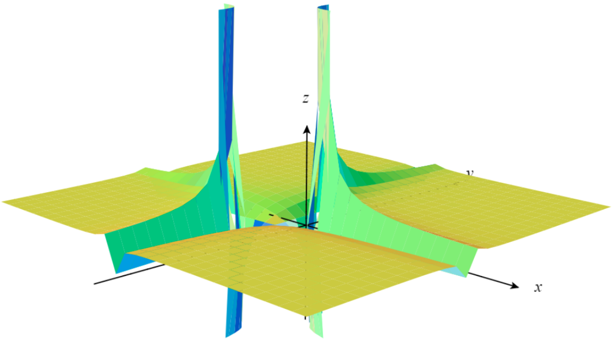


Figure 8: Broader view of long run behavior

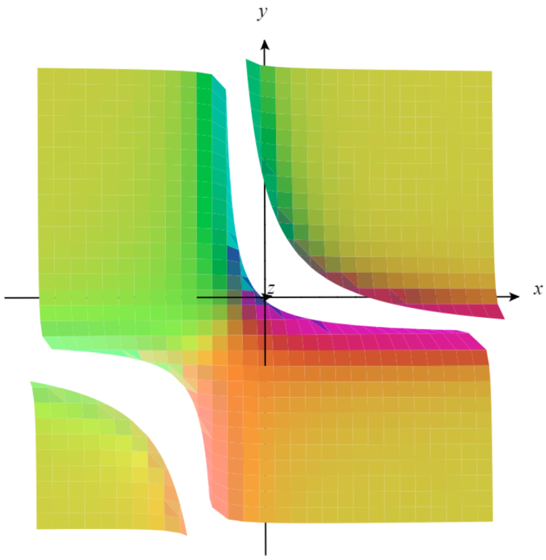


Figure 9: Top-down view of ratio plot

The function z also has vertical asymptotes, visible in Figure 9, which occur at any (ℓ, r) pair where the denominator of z goes to 0; that is, where

$$(\ell + 1)(r + 1) - \frac{3}{2} = 0 \tag{22}$$

which is equivalent to

$$\ell + r + r\ell = \frac{1}{2} \tag{23}$$

This equation divides the positive domain into two sections (Figure 10): the broader convergent section (pictured in blue), which satisfies the inequality

$$\ell + r + r\ell > \frac{1}{2} \tag{24}$$

and the divergent corner (grey), satisfying

$$\ell + r + r\ell \leq \frac{1}{2} \tag{25}$$

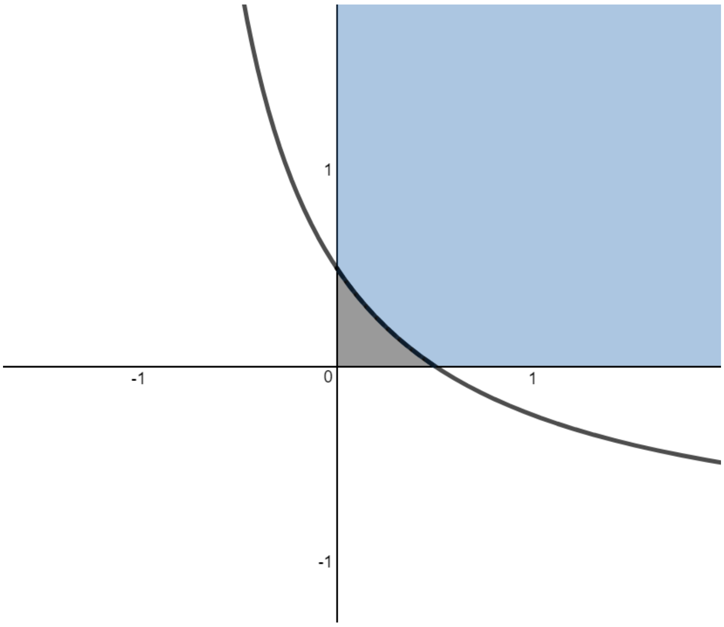


Figure 10: Partition of positive domain

It is certainly thrilling to think that this corner, intersecting the x and y axes at $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively, contains every positive (ℓ, r) pair that produces an ℓ-r flake of *divergent* area.

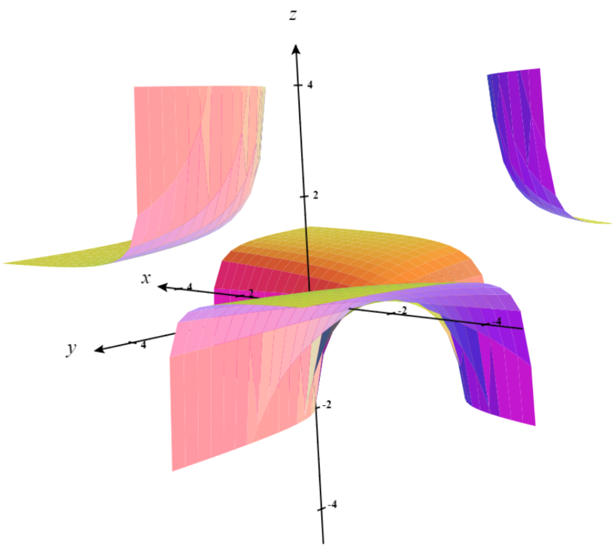


Figure 11: Alternative lateral view

6 Conclusion

By adding extended variability to the behavior initially described by the Koch Snowflake, we are better able to explore properties of these types of fractals, and examine patterns in the convergence of their areas.

Appendix A Condition of Convergence

Series converges only if denominator greater than numerator, that is, when the following is true:

$$\begin{aligned}(\ell + r + 1)^2 &> \ell^2 + 2 + r^2 \\ \ell^2 + r^2 + 1 + 2\ell + 2r + 2r\ell &> \ell^2 + 2 + r^2 \\ 2(\ell + r + r\ell + \frac{1}{2}) &> 2 \\ \ell + r + r\ell &> \frac{1}{2}\end{aligned} \tag{26}$$

Appendix B Convergence of Geometric Series

A geometric series of the form

$$\sum_{n=0}^{\infty} w^n \tag{27}$$

converges to

$$\frac{1}{1-w} \tag{28}$$

where $|w| < 1$.